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Quantum Theory of Measurement and the Modal Interpretations of Quantum Mechanics

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Several variants of the modal interpretation of quantum mechanics have been introduced and discussed in recent years. In this paper we present a study of the mathematical foundations of such an interpretation in the framework of the quantum theory of measurement.

1. INTRODUCTION

The empirical content of quantum mechanics is in the measurement outcome probabilities it predicts. The irreducible nature of these probabilities makes it difficult to develop a realistic interpretation of quantum mechanics, an interpretation which refers to individual objects and their properties. Recent advances in ultrahigh technology have made it obvious, however, that such an interpretation is—to say the least—desirable. Indeed, extremely controlled experimentation on individual objects, such as atoms, neutrons, electrons, and photons, is becoming a daily enterprise in experimental quantum physics.

There is now a growing literature on the so-called modal interpretations of quantum mechanics, several variants of which have already been put forward, for instance, in Bub (1992, 1994), Dieks (1989, 1994), Healey (1989), Kochen (1985), and van Fraassen (1991). They all aim to go beyond the purely statistical level of the description to provide a language which would come closer to the present-day experimental practice. This paper is devoted to a study of the measurement-theoretic content of these interpretations. (Further details and supplementary aspects are contained in the quoted papers of the authors).

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We begin in Section 2 with a brief review of some results on the problem of decomposability of mixed states into pure states and we present some facts concerning the ranges of states. We recall also the polar decomposition of an entangled vector state. In Section 3 these results are applied to define some sets of properties which can be and have been taken as the basis of formulating the corresponding modal interpretations of quantum mechanics. In Section 4 we formulate the conditions these interpretations posed on measurements, and in Section 5 we present a study of them within the theory of measurement. In this view the modal interpretations in question appear as particular specifications of the measurement process within the *minimal interpretation* of quantum mechanics.

The framework of this paper is the ordinary Hilbert space formulation of quantum mechanics, in which the description of a physical system is based on a complex separable Hilbert space $\mathcal{H}$, with the inner product $\langle \cdot | \cdot \rangle$. In their most common representation *states* and *observables* of the system are represented as density operators T and as self-adjoint operators A acting in $\mathcal{H}$, respectively. If $A = \int_{\mathbb{R}} a \, dP^A(a)$ is the spectral decomposition of A, then the *probability measure* defined by this observable and a state T obtains the explicit form $p_T^A(X) := \mathrm{tr}[TP^A(X)]$, where $P^A(X)$ is the spectral projection of A associated with the (Borel) subset X of the real line $\mathbf{R}$. In the *minimal interpretation*, these numbers are probabilities for measurement outcomes: $p_T^A(X)$ is the probability that a measurement of the observable A leads to a result in the set X when performed on the system in a state T. We recall further that for vector states $T = P[\varphi]$, generated by the unit vectors φ, these probabilities obtain the simple form $p_\varphi^A(X) = \langle \varphi | P^A(X)\varphi \rangle$.

2. PROPERTIES OF STATES

The decomposability properties of states as well as the properties of their ranges lead to several sets of properties which are at the heart of some of the modal interpretations of quantum mechanics. Therefore, we shall start by reviewing the basic properties of states.

2.1. Components of Mixed States

It is a basic fact of quantum mechanics that any state can be decomposed into vector states, e.g.,

$$T = \sum \lambda_i P[\psi_i] \tag{1}$$

but such a decomposition is never unique unless the state itself is a vector state. A given vector state $P[\varphi]$ is a *convex component* of a state T whenever

$$T = \lambda P[\varphi] + (1 - \lambda)T' \tag{2}$$

for some weight $0 \neq \lambda \leq 1$ and for some state T'. It is known that this is the case exactly when φ is in the range of the square root of T, that is, $\varphi \in \text{ran}(T^{1/2}) := \{T^{1/2}\varphi: \varphi \in \mathcal{H}\}$ (Hadjisavvas, 1981).

A decomposition (1) of a mixed state T into vector states $P[\psi_i]$ is *irreducible* if for each i, the vector ψ_i does not belong to the closure of the linear span of the other vectors $\{\psi_1, \ldots, \psi_{i-1}, \psi_{i+1}, \ldots\}$, that is,

$$\text{for all } i, \quad \psi_i \notin \overline{\text{lin}}\{\psi_1, \ldots, \psi_{i-1}, \psi_{i+1}, \ldots\} \tag{3}$$

We say that a vector state $P[\varphi]$ is an *irreducible convex component* of T if it participates in an irreducible decomposition of T. Again, it is known (Hadjisavvas, 1981) that this is the case exactly when φ is in the range of T, that is, $\varphi \in \text{ran}(T) := \{T\varphi: \varphi \in \mathcal{H}\}$. Clearly, if the decomposition (1) is orthogonal, that is, $\langle\psi_i|\psi_j\rangle = \delta_{ij}$, then it is also irreducible, but not necessarily the other way round.

2.2. The Polar Decomposition

The polar decomposition of an entangled vector state has a special role in some variants of the modal interpretation. Though this decomposition is well known, we need to recall it here. Therefore, let Ψ be a unit vector of the tensor product Hilbert space $\mathcal{H} \otimes \mathcal{K}$, and let $\{\varphi_i\}$ and $\{\phi_i\}$ be any orthonormal bases of $\mathcal{H}$ and $\mathcal{K}$, respectively. Then

$$\Psi = \sum \langle\varphi_i \otimes \phi_j|\Psi\rangle\varphi_i \otimes \phi_j \tag{4a}$$

and the partial traces (reduced states) $T(\Psi)$ and $W(\Psi)$ of $P[\Psi]$ over $\mathcal{K}$ and $\mathcal{H}$, respectively, obtain the decompositions into rank-one operators,

$$T(\Psi) = \sum \overline{c_{ij}}c_{kj}|\varphi_k\rangle\langle\varphi_i| \tag{4b}$$

$$W(\Psi) = \sum \overline{c_{ij}}c_{il}|\phi_l\rangle\langle\phi_j| \tag{4c}$$

where, for short, $c_{ij} = \langle\varphi_i \otimes \phi_j|\Psi\rangle$. The vector Ψ can be identified with a bounded linear map $F(\Psi)$: $\mathcal{K} \to \mathcal{H}$, which has the decomposition $F(\Psi) = \Sigma\, c_{ij}|\varphi_i\rangle\langle\phi_j|$ in terms of the given bases $\{\varphi_i\}$ and $\{\phi_i\}$. Let $F(\Psi) = UV$ be its *polar decomposition* (Reed and Simon, 1980), where V: $\mathcal{K} \to \mathcal{K}$ is a positive (compact) operator with the spectral structure $V = \Sigma \sqrt{v_i}P_i^V$, $\sqrt{v_i} > 0$, $\Sigma\, v_i = 1$, and U: $\mathcal{K} \to \mathcal{H}$ is a partial isometry, with $\ker(U)^\perp = \overline{\text{ran}}(V)$. Therefore, $F(\Psi) = \Sigma \sqrt{v_i}UP_i^V$, and if, for each i, $\{\psi_{ij}\}_j$ is an orthonormal basis of $P_i^V(\mathcal{K})$, then

$$F(\Psi) = \sum \sqrt{v_i} UP_i^V$$
$$= \sum_i \sqrt{v_i} \sum_j UP[\psi_{ij}] = \sum_i \sqrt{v_i} \sum_j |U\psi_{ij}\rangle\langle\psi_{ij}| \tag{5}$$

where $\langle U\psi_{ij} | U\psi_{ij'}\rangle = \delta_{jj'}$. Hence,

$$\Psi = \sum \sqrt{v_i} U\psi_{ij} \otimes \psi_{ij} \tag{6a}$$

$$T(\Psi) = \sum v_i P[U\psi_{ij}] = \sum v_i P_i^{T(\Psi)} \tag{6b}$$

$$W(\Psi) = \sum v_i P[\psi_{ij}] = \sum v_i P_i^V = \sum v_i P_i^{W(\Psi)} \tag{6c}$$

Clearly, the representation (6a) of the polar decomposition of Ψ depends on the choice of the vectors ψ_{ij}, being unique exactly when the eigenvalues of V are nondegenerate.

2.3. Ranges of States

For any state T, the following set inclusions hold true:

$$\mathrm{ran}(T) \subseteq \mathrm{ran}(T^{1/2}) \subseteq \overline{\mathrm{ran}}(T^{1/2}) = \overline{\mathrm{ran}}(T) \tag{7}$$

where, for instance, $\overline{\mathrm{ran}}(T^{1/2})$ denotes the closure of the range of the square root of T. These set inclusions are equalities if and only if the range of $T^{1/2}$ is closed. This is the case exactly when the range of T is finite dimensional.

The subspace inclusions (7) can be used to order states. Indeed, for any two states T_1 and T_2 we define

$$T_1 <_{\mathrm{cran}} T_2 \quad \text{iff} \quad \overline{\mathrm{ran}}(T_1) \subseteq \overline{\mathrm{ran}}(T_2) \tag{8a}$$

$$T_1 <_{\mathrm{rsq}} T_2 \quad \text{iff} \quad \mathrm{ran}(T_1^{1/2}) \subseteq \mathrm{ran}(T_2^{1/2}) \tag{8b}$$

$$T_1 <_{\mathrm{ran}} T_2 \quad \text{iff} \quad \mathrm{ran}(T_1) \subseteq \mathrm{ran}(T_2) \tag{8c}$$

where the abbreviations cran, rsq, and ran stand for the closure of the range, the range of the square root, and the range, respectively, which refer to the properties used to define these orders. Clearly, any of these relations defines a preorder on the set of states, the vector states being minimal with respect to each of them.

The spectral decompositions $T_1 = \sum t_{1,i} P_i^{T_1}$ and $T_2 = \sum t_{2,i} P_i^{T_2}$ lead still to another ordering of states:

$$T_1 <_{\mathrm{sd}} T_2 \quad \text{iff} \quad \text{for any } i, \quad P_i^{T_1} \leq P_j^{T_2}, \quad \text{for some } j \tag{9}$$

where sd stands for the spectral decomposition. Again, the vector states are minimal in this order, and $P[\varphi] <_{\mathrm{sd}} T$ iff $\varphi \in \mathrm{ran}(P_i^T)$ for some i.

3. SETS OF PROPERTIES

Let $\mathcal{P}(\mathcal{H})$ denote the set of projection operators on $\mathcal{H}$. Any state T is associated with two projection operators, its support projection P_T, and the complement of P_T, $N_T = I - P_T$. We recall that P_T is the smallest projection operator such that $T = TP_T = P_T T$, whereas N_T is the largest projection for which $N_T T = TN_T = O$.

For any state T we define the following three basic sets:

$$\mathcal{P}_1(T) := \{P \in \mathcal{P}(\mathcal{H}): \mathrm{tr}[TP] = 1\} = \{P \in \mathcal{P}(\mathcal{H}): P \geq P_T\} \tag{10}$$

$$\mathcal{P}_0(T) := \{P \in \mathcal{P}(\mathcal{H}): \mathrm{tr}[TP] = 0\} = \{P \in \mathcal{P}(\mathcal{H}): P \leq N_T\} \tag{11}$$

$$\mathcal{P}_{\neq 0}(T) := \{P \in \mathcal{P}(\mathcal{H}): \mathrm{tr}[TP] \neq 0\} = \{P \in \mathcal{P}(\mathcal{H}): P \wedge N_T \neq P\} \tag{12}$$

If $T = P[\varphi]$, we write, for instance, $\mathcal{P}_1(\varphi)$ instead of $\mathcal{P}_1(P[\varphi])$. Using the orderings (8) of states, we have, in addition,

$$\mathcal{P}_{\mathrm{cran}}(T) := \{P \in \mathcal{P}(\mathcal{H}): \text{there is a } T' <_{\mathrm{cran}} T \text{ such that } P \in \mathcal{P}_1(T')\} = \cup\{\mathcal{P}_1(\varphi): \varphi \in \overline{\mathrm{ran}}(T)\} \tag{13a}$$

$$\mathcal{P}_{\mathrm{rsq}}(T) := \{P \in \mathcal{P}(\mathcal{H}): \text{there is a } T' <_{\mathrm{rsq}} T \text{ such that } P \in \mathcal{P}_1(T')\} = \cup\{\mathcal{P}_1(\varphi): \varphi \in \mathrm{ran}(T^{1/2})\} \tag{13b}$$

$$\mathcal{P}_{\mathrm{ran}}(T) := \{P \in \mathcal{P}(\mathcal{H}): \text{there is a } T' <_{\mathrm{ran}} T \text{ such that } P \in \mathcal{P}_1(T')\} = \cup\{\mathcal{P}_1(\varphi): \varphi \in \mathrm{ran}(T)\} \tag{13c}$$

Similarly, the spectral ordering (10) allows one to define

$$\mathcal{P}_{\mathrm{sd}}(T) := \{P \in \mathcal{P}(\mathcal{H}): \text{there is a } T' <_{\mathrm{sd}} T \text{ such that } P \in \mathcal{P}_1(T')\} = \bigcup_i \cup\{\mathcal{P}_1(\varphi): \varphi \in \mathrm{ran}(P_i^T)\} \tag{13d}$$

Clearly, for any state T,

$$\mathcal{P}_1(T) \subseteq \mathcal{P}_{\mathrm{sd}}(T) \subseteq \mathcal{P}_{\mathrm{ran}}(T) \subseteq \mathcal{P}_{\mathrm{rsq}}(T) \subseteq \mathcal{P}_{\mathrm{cran}}(T) \subseteq \mathcal{P}_{\neq 0}(T) \tag{14}$$

On the basis of equation (13a) it is obvious that

$$\mathcal{P}_1(T) = \mathcal{P}_{\mathrm{cran}}(T) \Leftrightarrow T = P[\varphi] \qquad \text{for some unit vector } \varphi \tag{15}$$

On the other hand, one also proves that

$$\mathcal{P}_{\text{cran}}(T) = \mathcal{P}_{\neq 0}(T) \Leftrightarrow P_T = I \tag{16}$$

There are still two further subsets of projection operators which are of interest here. They arise from the polar and from an orthogonal decomposition of an entangled vector state Ψ of a compound system. If $F(\Psi) = \Sigma \sqrt{v_i} UP_i^V$ is the polar decomposition of Ψ, we define

$$\mathcal{P}_{\text{pd}}(\Psi) := \bigcup_i \{P \in \mathcal{P}(\mathcal{H} \otimes \mathcal{K})\text{: } P \geq P_i^{T(\Psi)} \otimes P_i^{W(\Psi)}\} \tag{17}$$

Let $\Psi = \Sigma \sqrt{v_i} U \psi_{ij} \otimes \psi_{ij}$ be a vector decomposition of the polar decomposition of Ψ. One may then also consider the set $\mathcal{P}_{\{\psi_{ij}\}}(\Psi)\text{: } = \cup \mathcal{P}_1(U\psi_{ij} \otimes \psi_{ij})$, which contains the set $\mathcal{P}_{\text{pd}}(\Psi)$. Apart from the arbitrariness of this set, it is to be noted that for each i, $P_i^{T(\Psi)} \otimes P_i^{W(\Psi)}$ is the smallest projection operator which contains all the projection operators $\Sigma\ P[U\psi_{ij}] \otimes P[\psi_{ij}]$, varying over the possible orthonormal bases $\{\psi_{ij}\}_j$ of $P_i^V(\mathcal{K})$. Therefore, it is the set (17) which shall be used subsequently. Clearly, if the eigenvalues v_i of $W(\Psi)$ are nondegenerate, then $\mathcal{P}_{\{\psi_i\}}(\Psi) = \mathcal{P}_{\text{pd}}(\Psi)$. We note also that $\mathcal{P}_1(\Psi) \subset \mathcal{P}_{\text{pd}}(\Psi) \subset \mathcal{P}_{\neq 0}(\Psi)$.

To introduce the other set, let

$$\mathcal{P}_{\text{obj}}(\Psi) := \{P \in \mathcal{P}(\mathcal{H} \otimes \mathcal{K})\text{: } PP[\Psi] = P[\Psi]P\}$$
$$= \mathcal{P}_1(\Psi) \cup \mathcal{P}_0(\Psi) \tag{18a}$$

If (R_i) is a sequence of mutually orthogonal projection operators such that $\Sigma\ R_i = I$, we write $\Psi = \Sigma\ R_i\Psi \equiv \Sigma\ \|R_i\Psi\|\Psi_i$, and define

$$\mathcal{P}_{\text{obj}}^{(R_i)}(\Psi) := \{P \in \mathcal{P}(\mathcal{H} \otimes \mathcal{K})\text{: } PP[\Psi_i] = P[\Psi_i]P \text{ for all } i\}$$
$$= \bigcap_i (\mathcal{P}_1(\Psi_i) \cup \mathcal{P}_0(\Psi_i)) \tag{18b}$$

Clearly, (18b) contains (18a) as a special case. By construction, it is obvious that the sets $\mathcal{P}_{\text{obj}}^{(R_i)}(\Psi)$ and $\mathcal{P}_{\neq 0}(\Psi)$ are incomparable.

In order to appreciate the content of the sets $\mathcal{P}_1(T)$, $\mathcal{P}_0(T)$, $\mathcal{P}_{\neq 0}(T)$, and $\mathcal{P}_{\text{Int}}(T)$, with Int = cran, rsq, ran, sd, pd, obj, we recall, first, that in quantum mechanics projection operators are commonly taken to describe properties of the system. Clearly, the set $\mathcal{P}_1(T)$ contains properties which the system *has* in state T in the sense of probability one, that is, a measurement of this property would show it with certainty. We recall also that a property P is *objective* in a state T if the system has this property or its complement property in that state. Accordingly, the set $\mathcal{P}_1(T) \cup \mathcal{P}_0(T)$ contains properties which are objective in the state T. In turn, the set $\mathcal{P}_{\neq 0}(T)$ contains the properties which are *possible* in the sense that their measurement outcome probabilities are nonzero.

The idea of the modal interpretations is that when a system is in a mixed state T or in an entangled state Ψ, then, in addition to the properties which the system has with certainty, it could also have some further properties. The specification of these properties is the task of such interpretations. The sets $\mathcal{P}_{\mathrm{Int}}(T)$ are proposed to do just that. They contain the properties which the system *could have* in state T according to the modal interpretation in question. For instance, the set $\mathcal{P}_{\mathrm{cran}}(T)$ contains all the properties which are possible in the state T, except those which are such that, whenever they are disjoint with the support projection P_T ($P \wedge P_T = O$), they are not orthogonal to it ($P \not\leq N_T$). We recall that P_T is the smallest property that the system has in that state. Since any property P which is compatible with P_T (that is, commutes with P_T), is orthogonal to P_T whenever it is disjoint with P_T, we may say, loosely speaking, that the set $\mathcal{P}_{\mathrm{cran}}(T)$ contains "all classically possible" properties in the state T. This does not mean, however, that the set $\mathcal{P}_{\mathrm{cran}}(T)$ was Boolean. The other options $\mathcal{P}_{\mathrm{Int}}(T)$ in (13) are further specifications of these "classical possibilities," whereas $\mathcal{P}_{\mathrm{pd}}(T)$ and $\mathcal{P}_{\mathrm{obj}}^{(R_i)}(T)$ are two alternative proposals for the sets of possible properties of the (compound) system in (an entangled) state $T = P[\psi]$. The set $\mathcal{P}_{\mathrm{obj}}(\Psi)$ contains properties which are objective in the vector state Ψ, whereas the set $\mathcal{P}_{\mathrm{obj}}^{(R_i)}(\Psi)$ is intended to represent the properties which *could be objective* together with the properties (R_i).

The Copenhagen variant of the modal interpretation of quantum mechanics (van Fraassen, 1991) takes the set $\mathcal{P}_{\mathrm{cran}}(T)$ as its basic ingredient, whereas the modal interpretations which rely on the polar decomposition theorem have the set $\mathcal{P}_{\mathrm{pd}}(\Psi)$, or $\mathcal{P}_{\mathrm{sd}}(T(\Psi))$ and $\mathcal{P}_{\mathrm{sd}}(W(\Psi))$, as the fundamental one (Kochen, 1985; Dieks, 1989, 1994; Healey, 1989). The set $\mathcal{P}_{\mathrm{obj}}^{(R_i)}(\Psi)$ is part of the basis of the modal interpretation proposed by Bub (1992, 1994); there this set is called the "fan" of the "handles" Ψ_i.

4. THE MODAL CONDITIONS ON MEASUREMENTS

The questions of the consistency and physical relevance of the various variants of the modal interpretation are most directly discussed in the framework of the quantum theory of measurement. To do that it is sufficient to consider discrete observables. Consider, therefore, an observable $A = \sum a_i P_i$. To model a measurement of A one usually fixes a measuring apparatus, with its Hilbert space $\mathcal{K}$, an initial state $P[\phi]$ of the apparatus, a pointer observable $Z = \sum z_i Z_i$, and a (unitary) measurement coupling $U: \mathcal{H} \otimes \mathcal{K} \to \mathcal{H} \otimes \mathcal{K}$. If $P[\varphi]$ is the initial state of the measured system, then $P[U(\varphi \otimes \phi)]$ is the system–apparatus state after the measurement. Denoting the corresponding reduced states of the measured system and the measuring apparatus as $T(U(\varphi \otimes \phi)) \equiv T(\varphi)$ and $W(U(\varphi \otimes \phi)) \equiv W(\varphi)$, respectively, we have the following schematic representation of the state transformations associated with a measurement:

$$\begin{array}{ccccc} P[\varphi] & & & & T(\varphi) \\ & \searrow & & \nearrow & \\ & & P[\varphi \otimes \phi] \rightarrow P[U(\varphi \otimes \phi)] & & \\ & \nearrow & & \searrow & \\ P[\phi] & & & & W(\varphi) \end{array}$$

A minimal requirement for $\mathcal{H}$, Z, ϕ, and U to constitute a measurement of A is the *calibration condition* (see, for instance, Busch *et al.*, 1991):

$$\text{for any } i \text{ and } \varphi, \quad \text{if } p^A_\varphi(a_i) = 1, \quad \text{then } p^Z_{W(\varphi)}(z_i) = 1 \tag{19a}$$

Using the notations of Section 3, we may express the calibration condition also as follows:

$$\text{for any } i \text{ and } \varphi, \quad \text{if } P_i \in \mathcal{P}_1(\varphi), \quad \text{then } Z_i \in \mathcal{P}_1(W(\varphi)) \tag{19b}$$

The calibration condition is equivalent to the apparently stronger *probability reproducibility condition*:

$$\text{for any } i \text{ and } \varphi, \quad p^A_\varphi(a_i) = p^Z_{W(\varphi)}(z_i) \tag{19c}$$

Using the terminology of Section 3, one may say that the basic requirement of a measurement is the following: *If a measurement of an observable is certain to yield a particular result, then the pointer observable has the corresponding value after the measurement.*

One of the basic questions of the modal interpretations is to find conditions under which the pointer observable could have the value z_i whenever the measurement outcome probability for a_i is nonzero. Extracting from the above quoted papers on the modal interpretations, these conditions read

$$p^A_\varphi(a_i) \neq 0 \Rightarrow Z_i \in \mathcal{P}_{\neq 0}(W(\varphi)) \tag{20a}$$

$$p^A_\varphi(a_i) \neq 0 \Rightarrow Z_i \in \mathcal{P}_{\text{cran}}(W(\varphi)) \tag{20b}$$

$$p^A_\varphi(a_i) \neq 0 \Rightarrow Z_i \in \mathcal{P}_{\text{rsq}}(W(\varphi)) \tag{20c}$$

$$p^A_\varphi(a_i) \neq 0 \Rightarrow Z_i \in \mathcal{P}_{\text{ran}}(W(\varphi)) \tag{20d}$$

$$p^A_\varphi(a_i) \neq 0 \Rightarrow Z_i \in \mathcal{P}_{\text{sd}}(W(\varphi)) \tag{20e}$$

$$p^A_\varphi(a_i) \neq 0 \Rightarrow I \otimes Z_i \in \mathcal{P}_{\text{pd}}(U(\varphi \otimes \phi)) \tag{20f}$$

$$p^A_\varphi(a_i) \neq 0 \Rightarrow I \otimes Z_i \in \mathcal{P}^{(I \otimes Z_i)}_{\text{obj}}(U(\varphi \otimes \phi)) \tag{20g}$$

We shall study next the implications of these "modal conditions" in a special measurement model.

5. THE MINIMAL MEASUREMENT MODEL

Consider, again, a discrete observable $A = \Sigma\, a_i P_i$. To determine the structure of a measurement $\langle \mathcal{K}, Z, \phi, U\rangle$ of A, we assume that the pointer observable is *minimal*. Any choice $Z = \Sigma\, z_i Z_i \equiv \Sigma\, z_i P[\phi_i]$ will then do, where $\{\phi_i\}$ is a basis of $\mathcal{K}$ (the dimension of $\mathcal{K}$ is thus fixed to equal the number of distinct eigenvalues of A). With that choice all the unitary mappings U which make $\langle \mathcal{K}, Z, \phi, U\rangle$ satisfy the calibration condition are completely characterized (Beltrametti *et al.*, 1990). To explicate these solutions, we assume here—for notational simplicity only—that all the eigenvalues of A are nondegenerate: $A = \Sigma\, a_i P_i = \Sigma\, a_i P[\varphi_i]$, where $\{\varphi_i\}$ is a complete orthonormal set of eigenvectors of A, $A\varphi_i = a_i\varphi_i$. If $\langle \mathcal{K}, Z, \phi, U\rangle$ is a measurement of A, then

$$\text{for each } \varphi_i, \quad U(\varphi_i \otimes \phi) = \gamma_i \otimes \phi_i \tag{21}$$

where γ_i are the unit vectors $\gamma_i = \Sigma_j \langle \varphi_j \otimes \phi_i | U(\varphi_i \otimes \phi)\rangle \varphi_j$. On the other hand, given any set of unit vectors $\{\gamma_i\}$ of $\mathcal{H}$, then (21) extends to a unitary mapping U such that $\langle \mathcal{K}, Z, \phi, U\rangle$ is a measurement of A. If $P[\varphi]$ is the initial state of the system, then the final states of the compound system, the measured system, and the apparatus are

$$U(\varphi \otimes \phi) = \sum \langle \varphi_i | \varphi\rangle \gamma_i \otimes \phi_i \tag{22a}$$

$$T(\varphi) = \sum |\langle \varphi | \varphi_i\rangle|^2 P[\gamma_i] \tag{22b}$$

$$W(\varphi) = \sum \langle \varphi_j | \varphi\rangle\langle \varphi | \varphi_i\rangle\langle \gamma_i | \gamma_j\rangle |\phi_j\rangle\langle \phi_i| \tag{22c}$$

We stress that in the case of a discrete observable with nondegenerate eigenvalues the *calibration condition poses no restrictions on the unit vectors* γ_i *which define the measurement coupling* U.

Consider a measurement of A given by the sequence of unit vectors $\{\gamma_i\}$. The minimal condition (19) on measurement leaves the set $\{\gamma_i\}$ completely *arbitrary*. But, clearly,

$$p^A_\varphi(a_i) \neq 0 \Rightarrow P[\phi_i] \in \mathcal{P}_{\neq 0}(W(\varphi)) \tag{23}$$

Further, it is immediate to observe that for any $P \in \mathcal{P}(\mathcal{K})$,

$$I \otimes P \in \mathcal{P}^{(\phi_i)}_{\text{obj}}(U(\varphi \otimes \phi)) \Leftrightarrow P \in \bigcap_i (\mathcal{P}_1(\phi_i) \cup \mathcal{P}_0(\phi_i)) \tag{24}$$

where, for short, we use the supindex (ϕ_i) instead of $(I \otimes P[\phi_i])$. This shows, in particular, that $I \otimes P[\phi_i] \in \mathcal{P}^{(\phi_i)}_{\text{obj}}(U(^\varphi \otimes \phi))$ whenever $p^A_\varphi(a_i) \neq 0$, that is, the modal requirement (20g) is *always* fulfilled.

No further conclusions on the validity of the modal conditions (21) can, in general, be drawn.

To discuss the solutions of the conditions (20) within the minimal measurement model, we shall assume that the vectors γ_i are such that $\overline{\text{lin}}\{\gamma_1, \ldots, \gamma_i, \ldots\} = \mathcal{H}$. This assumption simplifies the discussion, but implies, in fact, no loss of generality of our considerations.

We consider first the case that $\{\gamma_i\}$ is *linearly independent* in the algebraic sense. By a direct inspection one can see that the linear independence of the vectors γ_i is necessary for the condition (20b). If $\mathcal{H}$ is finite dimensional, it is also sufficient for that. However, in general, linear independence of the vectors γ_i is not enough for the modal conditions (20b)–(20d). Therefore, stronger requirements on $\{\gamma_i\}$ are to be posed.

The weakest possible "topological" strengthening of the linear independence of $\{\gamma_i\}$ is the ℓ_1-*linear independence*: for each sequence of complex numbers (a_i) such that $\Sigma\, |a_i| < \infty$, the condition $\Sigma\, a_i\gamma_i = 0$ implies that $a_i = 0$ for all i. Then we have $P[\phi_i] \in \mathcal{P}_{\text{cran}}(T)$ for each i and φ, such that $p^A_\varphi(a_i) \neq 0$, if and only if $\{\gamma_i\}$ is l_1-linearly independent (Cassinelli *et al.*, 1994).

We say that the sequence $\{\gamma_i\}$ is *irreducible* if, for each i, the vector γ_i is not contained in the set $\overline{\text{lin}}\{\gamma_1, \ldots, \gamma_{i-1}, \gamma_{i+1}, \ldots\}$. According to Section 2.1, this is the case exactly when the decomposition $T(\varphi) = \Sigma\, p^A_\varphi(a_i)P[\gamma_i]$ is irreducible. We then have $P[\phi_i] \in \mathcal{P}_{\text{rsq}}(W(\varphi))$ for each i and φ, such that $p^A_\varphi(a_i) \neq 0$, if and only if $T(\varphi) = \Sigma\, p^A_\varphi(a_i)P[\gamma_i]$ is an irreducible decomposition (Cassinelli *et al.*, 1994).

As a next step, we say that the sequence $\{\gamma_i\}$ has the *finiteness property* if it is linearly independent, and for each i, $\gamma_i = \theta_i + \Sigma^M_{j=1, j\neq i}\, a_j\gamma_j$, for some $\theta_i \in \overline{\text{lin}}\{\gamma_1, \ldots, \gamma_{i-1}, \gamma_{i+1}, \ldots\}^\perp$. The following result is obtained: $P[\phi_i] \in \mathcal{P}_{\text{ran}}(W(\varphi))$ for each i and φ, such that $p^A_\varphi(a_i) \neq 0$, if and only if $\{\gamma_i\}$ has the finiteness property (Cassinelli *et al.*, 1994).

Finally, we note that if $\{\gamma_i\}$ is an *orthonormal* sequence, then the decompositions (22) are just the polar and the spectral ones, in which case the conditions (20e) and (20f) are satisfied. Conversely, if either (20e) or (20f) is fulfilled for each i and φ, such that $p^A_\varphi(a_i) \neq 0$, then $\{\gamma_i\}$ is orthonormal (Lahti, 1990).

In conclusion, we have arrived at the following characterizations of the modal conditions (20) on measurements:

$$p^A_\varphi(a_i) \neq 0 \Rightarrow \quad Z_i \in \mathcal{P}_{\neq 0}(W(\varphi)) \Leftrightarrow \{\gamma_i\} \text{ arbitrary}$$

$$p^A_\varphi(a_i) \neq 0 \Rightarrow I \otimes Z_i \in \mathcal{P}^{(I\otimes Z_i)}_{\text{obj}}(U(\varphi \otimes \phi)) \Leftrightarrow \{\gamma_i\} \text{ arbitrary}$$

$$p^A_\varphi(a_i) \neq 0 \Rightarrow \quad Z_i \in \mathcal{P}_{\text{cran}}(W(\varphi)) \Leftrightarrow \{\gamma_i\}\ l_1\text{-independent}$$

$$p^A_\varphi(a_i) \neq 0 \Rightarrow \quad Z_i \in \mathcal{P}_{\text{rsq}}(W(\varphi)) \Leftrightarrow \{\gamma_i\} \text{ irreducible}$$

$$p^A_\varphi(a_i) \neq 0 \Rightarrow \quad Z_i \in \mathcal{P}_{\text{ran}}(W(\varphi)) \Leftrightarrow \{\gamma_i\} \text{ has finiteness}$$

$$p_\varphi^A(a_i) \neq 0 \Rightarrow \qquad Z_i \in \mathcal{P}_{sd}(W(\varphi)) \Leftrightarrow \{\gamma_i\} \text{ orthonormal}$$

$$p_\varphi^A(a_i) \neq 0 \Rightarrow \quad I \otimes Z_i \in \mathcal{P}_{pd}(U(\varphi \otimes \phi)) \Leftrightarrow \{\gamma_i\} \text{ orthonormal}$$

6. DISCUSSION

We have studied various sets of properties $\mathcal{P}_{\text{Int}}(T)$ which the system could have in state T according to the modal interpretation Int. We have characterized these sets in the context of the minimal measurement model where the state in question appears as the entangled state of the object–apparatus system after the measurement ($T = P[U(\varphi \otimes \phi)]$) or it is the mixed state of the apparatus after the measurement [$T = W(\varphi)$]. From the point of view of the quantum theory of measurement *the modal interpretations appear just as further specifications of a measurement process*; apart from Int = obj, they all structure the measurement beyond the calibration condition. It may be noted that the more liberal the modal interpretation is, in the sense of admitting a bigger set of properties which the apparatus could have after the measurement, the less restrictive it is from the point of view of the measurement theory.

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